THE REFLECTION OF WEAK CONVERGING SHOCK WAVES IN A GAS OF VARIABLE DENSITY

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Converging spherical, cylindrical and plane shock waves in a variable-density gas have already been discussed in [1]. The initial (undisturbed) condition of the ideal gas was given in the form

$$p \equiv p_0, \qquad \rho = \omega r^s, \qquad v \equiv 0 \tag{1}$$

where r, p, ρ, v are, respectively, the distance of particles from the center (or plane) of symmetry, the pressure, density and mass velocity, whilst p_0, s and ω are constants. A converging shock wave propagates through the gas in state (1) according to a power law in terms of time. It was demonstrated in [1] that, depending on the quantities s and ν ($\nu = 1$, 2, 3 for plane, cylindrical and spherical waves, respectively), there occur qualitatively differing cases of behavior of the converging waves when $r \to 0$.

For $s < 2(\nu - 1)$, shock waves increase in intensity without limit, or reach a constant intensity when $r \to 0$; in that case it is possible to construct self-similar solutions with implosion waves which possess these features. An analysis of such self-similar solutions reveals that after reflection from the center (axis) of a constant-intensity wave the gas behind the reflected front comes to rest at constant pressure.

When $s > 2(\nu - 1)$, there are no longer any such self-similar solutions. For such cases in [1], solutions of the linearized gasdynamic equations are constructed, which describe the motion of a gas in the case of a weak converging shock wave. An analysis of the solutions has revealed that the conditions of linearization are fulfilled when $r \to 0$, $t \to 0$, where $s > 2(\nu - 1)$; in particular, the intensity of the converging waves tends to 0 when $r \to 0$.

However, according to the linear solution, both pressure and particle velocity increase without limit close to the front. Linear equations should therefore not be used in the neighborhood of the reflected front where nonlinear terms are so important.

In this paper we construct a solution which takes account of the nonlinear terms and is valid asymptotically (for $r \to 0$), and is also valid close to the reflected front. To do this, use is made of the simultaneous expansion in series of both dependent and independent variables, which is the method described in [2] due to Poincaré, Lighthill and Go.

Landau [3] was the first to study nonlinear effects in weak diverging waves, whilst Gubkin [4] deals with a gas of variable density.

The velocity, pressure and density of the gas before the disturbance starts are assumed to be given by Formula (1), where $s > 2(\nu - 1)$ and $u(r, t)$ is the particle displacement reckoned from the initial condition. The equations for the one-dimensional adiabatic motion of an ideal gas can be taken in the form (r is a Lagrange coordinate)

$$\rho (r + u)^{\nu-1}\left(1 + \frac{\partial u}{\partial r}\right) = \omega r^s r^{\nu-1}, \qquad \left(1 + \frac{\partial u}{\partial r}\right)\rho \frac{\partial^2 u}{\partial t^2} + \frac{\partial p}{\partial r} = 0 \tag{2}$$

$$p\rho^{-\gamma} = p_0 (\omega r^s)^{-\gamma}$$

The first of the two equations of system (2) express the conservation of mass and of impulse, whilst the third expresses the conservation of particle entropy (γ is the adiabatic index). In general, the latter condition does not hold in the presence of shock waves, but it can be assumed to be approximately valid for weak waves, for the jump in entropy in them is a third-order quantity as compared with the rise in velocity or pressure. In the solution which will be constructed here, the shock waves are weak, so that the assumption can be justified.

The first and third equations of system (2) allow us to express p and ρ in terms of u:

$$p = p_0 \left(1 + \frac{\partial u}{\partial r}\right)^{-\gamma} \left(1 + \frac{u}{r}\right)^{-\gamma(\nu-1)}, \qquad \rho = \omega r^s \left(1 + \frac{\partial u}{\partial r}\right)^{-1} \left(1 + \frac{u}{r}\right)^{1-\nu} \tag{3}$$

If we substitute Equations (3) in the momentum equation (2), we find an equation for the function $u(r, t)$:

$$\frac{\omega r^s}{\gamma p_0}\left(1 + \frac{\partial u}{\partial r}\right)^{\gamma+1} \left(1 + \frac{u}{r}\right)^{(\gamma-1)(\nu-1)} \frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial r^2} + \frac{\nu - 1}{r + u}\left(1 + \frac{\partial u}{\partial r}\right)\left(\frac{\partial u}{\partial r} - \frac{u}{r}\right) \tag{4}$$

The characteristics of this nonlinear hyperbolic second-order equation are determined by the equation

$$\frac{dt}{dr} = \pm \sqrt{\frac{\omega r^s}{\gamma p_0}\left(1 + \frac{\partial u}{\partial r}\right)^{(\gamma+1)/2} \left(1 + \frac{u}{r}\right)^{[(\gamma-1)(\nu-1)]/2}} \tag{5}$$

If we linearize Equation (4), assuming $u \ll r$ and $\partial u/\partial r \ll 1$, its characteristics near $r = 0$, $t = 0$, as follows from (5), will be the parabolas

$$t = \mp \frac{2}{2+s}\sqrt{\frac{\omega}{\gamma p_0}}\, r^{(2+s)/2}$$

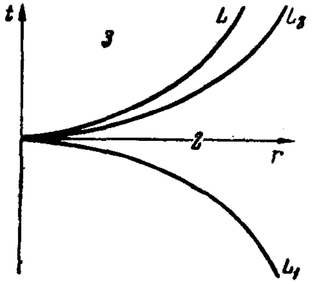

Fig. 1.

These are shown in Fig. 1 as curves L_1 and L_2 in the rt-plane and represent fronts of converging and of reflected waves in the linear approximation. Instead of t, now, we introduce a new dimensionless variable x:

$$x = \frac{1}{2}\left[\frac{(2+s)\sqrt{\gamma p_0}}{2\sqrt{\omega}}\, t r^{-(2+s)/2} + 1\right] \tag{6}$$

It is evident that $x = 0$ on curve L_1, and $x = 1$ on curve L_2. We will assume u to be a function of x and of r, and we will denote $\partial u/\partial x(x, r)$ by $z(x, r)$. Going over to the new variables, now, instead of Equations (4), we arrive at a system of two equations for the functions $u(x, r)$ and $z(x, r)$:

$$\frac{\partial u}{\partial x} - z = 0 \tag{7}$$

$$\begin{aligned}&\left\{x(1-x) + \frac{1}{4}\left[\left(1 + \frac{\partial u}{\partial r} + \frac{2+s}{4}\frac{1-2x}{r}z\right)^{\gamma+1}\left(1+\frac{u}{r}\right)^{(\gamma-1)(\nu-1)} - 1\right]\right\}\frac{\partial z}{\partial x} - \\ &- \frac{4r^2}{(2+s)^2}\frac{\partial^2 u}{\partial r^2} - \frac{2(1-2x)}{2+s}r\frac{\partial z}{\partial r} + \frac{6+s-2\nu}{2(2+s)}(1-2x)z - \\ &- \frac{4(\nu-1)}{(2+s)^2}\left(r\frac{\partial u}{\partial r} - u\right) - \frac{\nu-1}{r+u}\left[\frac{2}{2+s}\left(r\frac{\partial u}{\partial r} - u\right) + \frac{1-2x}{2}z\right]^2 = 0\end{aligned}$$

In order to solve system (7) we formally expand the required functions and one of the independent variables in a series of powers of the other independent variable, i.e. we obtain

$$\begin{aligned} u &= C\eta^{1+k}w^{(0)}(\xi) + C^2\eta^{1+2k}w^{(1)}(\xi) + \cdots \\ z &= C\eta^{1+k}q^{(0)}(\xi) + C^2\eta^{1+2k}q^{(1)}(\xi) + \cdots \\ x &= \xi \qquad\qquad\quad + C\,\eta^{k}\,\varphi^{(1)}(\xi) + \cdots \\ r &= \eta \end{aligned} \tag{8}$$

In the expansions (8), C is an arbitrary constant of dimensions (cm^{-k}), whilst all the $w^{(n)}$, $q^{(n)}$, $\phi^{(n)}$ are dimensionless functions of the new dimensionless argument ξ. From now on we will drop the indices of the functions $w^{(0)}$, $q^{(0)}$ and $\phi^{(1)}$, i.e. $w^{(0)} = w$, $q^{(0)} = q$, $\phi^{(1)} = \phi$.

If we differentiate the expansion for x with respect to x and r, we find the derivatives

$$\frac{\partial\xi}{\partial x} = 1 - C\eta^k\varphi'(\xi) - \cdots, \qquad \frac{\partial\xi}{\partial r} = -Ck\eta^{k-1}\varphi(\xi) - \cdots$$

With the help of these equations we express the derivatives with respect to x, r, in terms of derivatives with respect to the new variables ξ, η. We then obtain (strokes denoting differentiation with respect to ξ)

$$\frac{\partial}{\partial x} = (1 - C\eta^k\varphi' - \cdots)\frac{\partial}{\partial\xi}, \qquad \frac{\partial}{\partial r} = \frac{\partial}{\partial\eta} - (Ck\eta^{k-1}\varphi + \cdots)\frac{\partial}{\partial\xi} \tag{9}$$

On transforming system (7), using (8) and (9), i.e. on going over to the new variables, and on expanding in powers of η, we find that the left-hand sides of Equations (7) are power series in η of the type

$$\sum_{n=0}^{\infty} a^{(n)}(\xi)\,\eta^{1+(n+1)k}$$

If we equate coefficients of successive powers of η to zero, we arrive at ordinary differential equations for functions $w^{(n)}$, $q^{(n)}$. The system of equations for the nth approximation will be of the form

$$\begin{gathered}\frac{dw^{(n)}}{d\xi} - q^{(n)} = f^{(n)}\\ \xi(1-\xi)\frac{dq^{(n)}}{d\xi} + (1-2\xi)\left[\frac{1}{2} - \frac{2(n+1)k+\nu}{2+s}\right]q^{(n)} - \\ - \frac{4k(n+1)[(n+1)k+\nu]}{(2+s)^2}\,w^{(n)} = g^{(n)}\end{gathered} \tag{10}$$

This system is linear as to the functions $w^{(n)}$ and $q^{(n)}$; the right-hand side of the first equation $f^{(n)}$ depends only on $w^{(i)\prime}$, $\phi^{(j)\prime}$, and, further, $g^{(n)}$ is a function also of $w^{(i)}$, $\phi^{(j)}$, $q^{(i)}$, $q^{(i)\prime}$ $(0 \leqslant i \leqslant n-1,\ 1 \leqslant j \leqslant n)$ and explicitly a function of ξ. The functions $\phi^{(n)}(\xi)$ can be chosen arbitrarily.

For the zero approximation $f^{(0)} = g^{(0)} \equiv 0$, and therefore $q = w'$. For the first approximation system $(n = 1)$, on carrying out the given expansion in η, we find

$$f^{(1)} = w'\varphi', \qquad g^{(1)} = P + Q\varphi + R\varphi' \tag{11}$$

Here P, Q, R are definite functions of ξ, w, w' and w'' (in this case $q = w'$); they are

$$\begin{gathered}P = \frac{\gamma+1}{4}\frac{2+s}{4}(2\xi-1)w'w'' - \frac{1}{4}[(\gamma-1)(\nu-1) + (\gamma+1)(1+k)]ww'' + \\ + (\nu-1)\left[\frac{2k}{2+s}w + \frac{1-2\xi}{2}w'\right]^2\end{gathered} \tag{12}$$

$$Q = \left(1 + \frac{2k}{2+s}\right)\left[(2\xi - 1)\, w'' + \left(1 - \frac{6k+2\nu}{2+s}\right) w'\right], \qquad R = \xi\,(1-\xi)\, w''$$

Evidently the zero approximation for the solution of the nonlinear equations, in which the following are assumed

$$u = C\eta^{1+k} w(\xi), \qquad z = C\eta^{1+k} q(\xi), \qquad x = \xi, \qquad r = \eta$$

is in fact the exact solution of the linearized equations. Such a linear solution, which represents the reflection of a converging wave, has been constructed in [1]. The function $w(\xi)$ satisfies the hypergeometric equation derived from system (10) for $n = 0$.

Before the advent of the disturbance ($\xi \leqslant 0$), we have $w = q \equiv 0$ within the region 1 of Fig. 1. The solution in region 2 ($0 < \xi < 1$) satisfies initial conditions $w(0) = 0$, $w'(0) = 1$, which means that at the front of a converging wave the motion is continuous, whilst pressure and velocity experience a finite jump. These conditions can only be fulfilled when

$$k = \frac{2+s}{4} - \frac{\nu}{2} > 0 \qquad (s > 2(\nu - 1)) \tag{13}$$

which fixes k.

In region (3) ($1 \leqslant \xi \leqslant \infty$) the solution should satisfy boundary conditions expressing the absence of any source or sink when $r = 0$ ($v(0,t) = 0$) and the continuity of motion on the reflected front.

Under these conditions the solution of the zero approximation has the following form (subscript denotes number of region):

$$\begin{gathered} w_2 = \xi F(1-\varepsilon,\ \varepsilon;\ 2;\ \xi), \qquad q_2 = F(1-\varepsilon,\ \varepsilon;\ 1;\ \xi) \\ w_3 = \frac{\Gamma(1-\varepsilon)}{\Gamma(1+\varepsilon)\,\Gamma(2-2\varepsilon)}\, \xi^{\varepsilon} F(-\varepsilon,\ 1-\varepsilon;\ 2-2\varepsilon;\ \xi^{-1}) \\ q_3 = \frac{\Gamma(1-\varepsilon)}{\Gamma(\varepsilon)\,\Gamma(2-2\varepsilon)}\, \xi^{\varepsilon-1} F(1-\varepsilon,\ 1-\varepsilon;\ 2-2\varepsilon;\ \xi^{-1}) \\ \varepsilon = \frac{1}{2} - \frac{\nu}{2+s} > 0 \end{gathered} \tag{14}$$

where F is a hypergeometric series.

In the solution under discussion it follows from (3), (6) and (8) that displacement, velocity and pressure can be expressed in terms of $w(\xi)$ in the following manner:

$$u = C\eta^{1+k}w, \qquad v = \frac{\partial u(r, t)}{\partial t} = C\frac{2+s}{4}\sqrt{\frac{\gamma p_0}{\omega}}\,\eta^{k-s/2}w'$$
$$p - p_0 = -\gamma p_0 C\frac{2+s}{4}\eta^k\left[(1-2\xi)w' + \left(1+\frac{2\nu}{2+s}\right)w\right] \tag{15}$$

Let us calculate the pressure jump $(\Delta p)_0$ at the front of a converging wave (for $\xi = 0$). Bearing in mind that $w(0) = 0$, $w'(0) = 1$, we find from (15) that*

$$(\Delta p)_0 = -\gamma p_0 C\frac{2+s}{4}\eta^k \tag{16}$$

Owing to the fact that compression takes place in the shock wave and $(\Delta p)_0 > 0$, we have to assume that $C < 0$.

We look for an expansion of solution (14) in the neighborhood of the point $\xi = 1$ (the front of the reflected wave). To do this we use the formula [5] for the analytic continuation of the hypergeometric series with a special relation between its parameters

$$F(\alpha, \beta; \alpha+\beta+m; z) = \frac{\Gamma(m)\Gamma(\alpha+\beta+m)}{\Gamma(\alpha+m)\Gamma(\beta+m)}\sum_{n=0}^{m-1}\frac{(\alpha)_n(\beta)_n}{(1-m)_n n!}(1-z)^n +$$
$$+\frac{(z-1)^m\Gamma(\alpha+\beta+m)}{\Gamma(\alpha)\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{(\alpha+m)_n(\beta+m)_n}{(n+m)!\,n!}[h_n - \ln(1-z)](1-z)^n \tag{17}$$
$$(-\pi < \arg(1-z) < \pi;\ \alpha, \beta \neq 0, -1, \ldots;\ m = 0, 1, 2, \ldots)$$

The following notation has been used in (17):

$$h_n = \psi(n+1) + \psi(n+m+1) - \psi(\alpha+n+m) - \psi(\beta+n+m)$$
$$\psi(x) = d\ln\Gamma(x)/dx, \qquad (\alpha)_n = \alpha(\alpha+1)\ldots(\alpha+n-1), \qquad (\alpha)_0 = 1$$

whilst when $m = 0$ the first term on the right-hand side of (17) should be considered equal to zero.

If we write down the hypergeometric series (14) according to Formula (17), we can find the required expansions

* The formula in geometric acoustics; $\Delta p = \text{const}\ (\rho c r^{1-\nu})^{1/2}$, where c is the velocity of sound, also leads to the same law of pressure change at a converging wave front (for the same value of k).

$$
\begin{aligned}
w_{2,3} &= AB^{-1} + A(1-\xi)\ln|1-\xi| + O(1-\xi) \qquad (18)\\
w_{2,3}' = q_{2,3} &= -A\ln|1-\xi| - AD_{2,3} + O[(1-\xi)\ln|1-\xi|]\\
w_{2,3}'' &= A(1-\xi)^{-1} + AB\ln|1-\xi| + O(1)
\end{aligned}
$$

In these relations (18), the coefficient D_2 should apply to region 2, D_3 to 3, whilst A and B are the same for both regions and are given by

$$
\begin{aligned}
&A = \frac{1}{\pi}\cos\frac{\pi\nu}{2+s} > 0, \qquad B = \frac{1}{4} - \left(\frac{\nu}{2+s}\right)^2 > 0 \qquad (19)\\
&D_2 = 2E + \psi\left(\frac{1}{2} - \frac{\nu}{2+s}\right) + \psi\left(\frac{1}{2} + \frac{\nu}{2+s}\right), \qquad D_3 = 2E + 2\psi\left(\frac{1}{2} + \frac{\nu}{2+s}\right)
\end{aligned}
$$

Here E = 0.577, which is Euler's constant. From a well-known identity [6] for the logarithmic derivative of Euler's Gamma Function $\psi(x)$, we have

$$
D_3 - D_2 = \psi\left(\frac{1}{2} + \frac{\nu}{2+s}\right) - \psi\left(\frac{1}{2} - \frac{\nu}{2+s}\right) = \pi \quad \frac{\pi\nu}{2+s} \qquad (20)
$$

Substitute the expansion (18) in Formula (12) to find the expansions of functions P, Q, R near the point $\xi = 1$:

$$
\begin{aligned}
&P = -\frac{(\gamma+1)(2+s)}{16} A^2 \frac{\ln|1-\xi|}{1-\xi} + \frac{G}{1-\xi} + O(\ln^2|1-\xi|) \qquad (21)\\
&Q = \left(\frac{3}{2} - \frac{\nu}{2+s}\right)\frac{A}{1-\xi} + O(\ln|1-\xi|), \qquad R = A + O[(1-\xi)\ln|1-\xi|]\\
&G_{2,3} = -\frac{(\gamma+1)(s+2)}{16} A^2 D_{2,3} - \left[(\gamma-1)(\nu-1) + (\gamma+1)\frac{6+s-2\nu}{4}\right]\frac{A^2}{4B}
\end{aligned}
$$

It is evident from the last equation that the coefficient G has different values in regions 2 and 3.

The linear solution constructed above embodies, for $\xi = 1$, a singularity which should not occur with an exact solution of the nonlinear equations and which leads to infinite velocity and pressure at the reflected front. It is evident from Equation (10) that the functions $w^{(n)}$, $q^{(n)}(n = 1, 2, \ldots)$, in general, also have singularities at the point $\xi = 1$, and therefore the above approximations might worsen the solution of the zero approximation. The idea of the Poincaré-Lighthill-Go method is to lower the order of singularities by suitable choice of the function $\phi^{(n)}$.

The right-hand side $g^{(1)}$ of the first approximation equation ($n = 1$) has, at $\phi(\xi) = 0$, a singularity at the point $\xi = 1$, the character of which is defined by the expansion P in Formulas (21). It can be lowered, as seen from (11), (12) and (21), by taking the function ϕ in regions 2 and 3 equal to

$$\varphi_{2,3}(\xi) = -\Phi A^{-1}w'_{2,3} + BA^{-1}(T_{2,3} - \Phi D_{2,3})\,w_{2,3}$$
$$= \Phi \ln|1-\xi| + T_{2,3} + O\,[(1-\xi)\ln|1-\xi|] \qquad (22)$$

The constants Φ, T_2 and T_3 are determined from the condition that the coefficients of the two first terms in the expansion $g^{(1)}$ vanish, i.e. terms of the order $(1-\xi)^{-1}\log(1-\xi)$ and $(1-\xi)^{-1}$. We thus obtain

$$\Phi = \frac{(\gamma+1)(s+2)^2 A}{8(6+3s-2\nu)} > 0, \qquad T_{2,3} = \frac{(\gamma+1)(s+2)^3 A}{4(6+3s-2\nu)^2} - \frac{2(2+s)\,G_{2,3}}{(6+3s-2\nu)A} \quad (23)$$

If ϕ is given by Formulas (22), (23), then the functions $f^{(1)}$, $g^{(1)}$ will be of order $f^{(1)} = O(1-\xi)^{-1}\log(1-\xi)$, $g^{(1)} = O\log^2(1-\xi)$, whilst the solutions to the first approximation equation $w^{(1)}$, $q^{(1)}$ are of order $w^{(1)} = O\log^2(1-\xi)$, $q^{(1)} = O\log^2(1-\xi)$. Hence the ratio between the second and the zero approximations in expansion (8) is

$$\frac{C^2\eta^{1+2k}w^{(1)}}{C\eta^{1+k}w} = \eta^k O(\ln^2|1-\xi|), \qquad \frac{C^2\eta^{1+2k}q^{(1)}}{C\eta^{1+k}q} = \eta^k O(\ln|1-\xi|)$$

We will demonstrate below that the solutions do not have a physical meaning for all values of ξ, indeed only for $|1-\xi| > a\eta^k$ (where a is a positive constant), and therefore these ratios tend to zero for $\eta \to 0$ even on the reflected front.

In the following, therefore, we will confine ourselves to the first term only in the expansions (8) for u and z, whilst for x we will use the first two terms. In this approximation the solution is constructed already, because functions w, q, and ϕ have been found. When the difference $(1-\xi)$ differs significantly from zero, the solution hardly differs from the linear one for $\eta \to 0$, for x is very close to ξ, whilst functions w and q are discontinuous for $\xi \neq 1$.

The connection between x and ξ for ξ close to unity is expressed by the following formula, derived from (8) and (22):

$$1 - x = 1 - \xi - C\eta^k(\Phi\ln|1-\xi| + T_{2,3}) \qquad (24)$$

Here, for $\xi < 1$ one should take T_2 and for $\xi > 1$, T_3. The relation between $(1-x)$ and $(1-\xi)$ for small fixed values of η is shown in Fig. 2, where C is assumed negative; for C positive the direction of the coordinate axes in Fig. 2 should be changed. The magnitude of $(1-x)$ has a maximum for $1-\xi = C\Phi\eta^k < 0$, and it tends to $-\infty$ when $\xi \to 1$.

In a solution which has physical meaning, ξ should be a single-valued function of x and η (i.e. r and t). Otherwise, w and q, which depend on ξ, will also not be single-valued functions of x and r, which, in view of Formula (15), leads to multiple-valued displacement, velocity and pressure as functions of the coordinates and of time.

In order to ensure a single-valued relationship between ξ and x it is necessary to pass from one branch of the curve (24) to the other, as is shown in Fig. 2. In this case, $1 - \xi$ will be a monotonically increasing discontinuous (with a break) function of $1 - x$; when $x = x_*$ it has a jump from $1 - \xi_3 < 0$ to $1 - \xi_2$, whilst it does not have any values between $1 - \xi_3$ and $1 - \xi_2$. It is obvious that for a single-valued relation between $1 - \xi$ and $1 - x$ the following condition should be fulfilled:

$$1 - \xi_3 < C\Phi\eta^k$$

The position of the jump is completely defined by the magnitudes ξ_2 and ξ_3; they, just like the whole curve in Fig. 2, depend on η. One (algebraic) equation connecting ξ_2 and ξ_3 will be found from the condition that $1 - x$ has one and the same value as $1 - x_*$ when $\xi = \xi_2$ and $\xi = \xi_3$. From (24) we get

$$(1-\xi_2)-(1-\xi_3)-C\eta^k\left[\Phi\ln\left|\frac{1-\xi_2}{1-\xi_3}\right|+(T_2-T_3)\right]=0 \qquad (25)$$

To find the second equation we notice that the discontinuity in the function $\xi(x, \eta)$ denotes a jump in the physical quantities on the reflected front. On substituting expansion (18) in Formula (15), one can see that the conditions of continuity of displacement and conservation of momentum at the front of the reflected wave, i.e.

$$u_2 = u_3, \qquad p_2 - p_3 = \sqrt{\gamma p_0 \omega r^s}\,(v_2 - v_3)$$

are automatically fulfilled with our approximation.

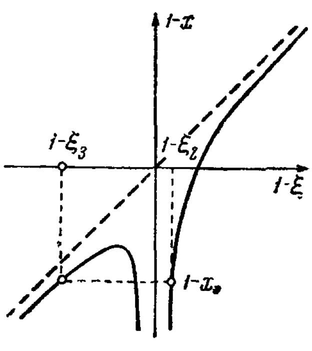

Fig. 2.

A further condition should be fulfilled on the shock wave, which can be formulated (for a weak shock) in the following manner: angles made by the trajectories of a wave in the rt-plane with the characteristics are equal on both sides of the wave [7]. If $t(r)$ is the trajectory of the reflected front, we obtain, approximately, from (5)

$$\frac{dt_*}{dr}=\sqrt{\frac{\omega r^s}{\gamma p_0}}\left\{1+\frac{\gamma+1}{4}\left[\frac{\partial u_2(r,\ t)}{\partial r}+\frac{\partial u_3(r,\ t)}{\partial r}\right]+\right.$$
$$\left.+\frac{(\gamma-1)(\nu-1)}{4r}\left[u_2(r,\ t)+u_3(r,\ t)\right]\right\}$$

In this equation we transform to the variables x, r and express t through x by using (6). We then insert the expansion (8), limiting ourselves to the approximations taken there. The functions w and w' of the arguments ξ_2 and ξ_3 can be represented in the form (18), retaining only the main terms, whilst we will express x through η and $\xi_2(\eta)$ by Formula (24). A differential equation connecting $\xi_2(\eta)$ and $\xi_3(\eta)$ will be obtained:

$$-2(1-\xi_2)+2\left(1+\frac{2k}{2+s}\right)(\Phi\ln|1-\xi_2|+T_2)C\eta^k+$$
$$\frac{4\eta}{2+s}\left(\frac{C\Phi\eta^k}{1-\xi_2}-1\right)\frac{d(1-\xi_2)}{d\eta}=\left[(\gamma-1)(\nu-1)+\frac{\gamma+1}{4}(6+s-2\nu)\right]\frac{A}{2B}C\eta^k+$$
$$+\frac{(\gamma+1)(s+2)}{16}C\eta^kA\left[\ln|1-\xi_2|+\ln|1-\xi_3|+D_2+D_3\right] \qquad (26)$$

We will seek a solution of the systems (25), (26) in the form

$$1-\xi_2=-\lambda C\Phi\eta^k,\qquad 1-\xi_3=\mu C\Phi\eta^k,\qquad \lambda>0,\qquad \mu>0\qquad (C<0)\quad (27)$$

where λ, μ are constants; here too, the natural initial condition $x = 1$ for $\eta = 0$ is fulfilled, i.e. the jump in the rt-plane is tangential to the characteristic of the linear equation.

If the substitution (27) is carried out, terms of the type $\eta^k \log\eta$ in Equation (26) will cancel, and after cutting off at η^k the relations (25) and (26) will yield a system of two transcendental equations for the quantities λ and μ (in the transformations we make use of the values of the constants from Formulas (19), (20), (21) and (23)):

$$\lambda+\mu+\ln\frac{\lambda}{\mu}-\pi\tan\frac{\pi\nu}{2+s}=0,\ 2\lambda+2+\ln\frac{\lambda}{\mu}-\pi\tan\frac{\pi\nu}{2+s}=0$$

From this, $\lambda = \mu - 2$, whilst for μ we arrive at the equation

$$2\mu-2+\ln\frac{\mu-2}{\mu}-\pi\tan\frac{\pi\nu}{2+s}=0 \qquad (28)$$

The left-hand side of Equation (28) at $\mu > 2$ is a monotonically increasing function of μ, tending to $-\infty$ when $\mu \to 2+0$ and to $+\infty$ when $\mu \to +\infty$. Consequently, Equation (28) has the one root $\mu > 2$. Here, $1-\xi_3 < C\Phi\eta^k$, $1-\xi_2 > 0$, as it should be. The value of μ can be found easily for given values of ν and s.

Thus, all solutions, including the reflected wave front, are fully constructed in this approximation. The position of the reflected front is determined by Formulas (27); if we substitute these into Equations (24) and (6), we find the more accurate trajectory of the discontinuity in the rt-plane

$$t_* = \frac{2\sqrt{\omega}}{(2+s)\sqrt{\gamma p_0}} r^{(2+s)/2}[1 + 2Ck\Phi r^k \ln r + O(r^k)]$$

It is represented by the curve L in Fig. 1.

From Equations (27) and (15) we obtain the displacement, velocity and pressure close to the reflected front

$$u = CAB^{-1}r^{1+k} + O(r^{1+2k}\ln r)$$

$$v = -C\frac{2+s}{4}\sqrt{\frac{\gamma p_0}{\omega}}Akr^{k-s/2}\ln r + O(r^{k-s/2})$$

$$p - p_0 = -\gamma p_0 C\frac{2+s}{4}Akr^k \ln r + O(r^k)$$

Using available formulas it is possible to find also the following terms (not written down at first) of the expansion of t_*, v, and p into which the value of μ enters.

The pressure jump at the front of the reflected wave, according to (15), is equal in the first approximation

$$\Delta p = p_3 - p_2 = -\gamma p_0 C\frac{2+s}{4}r^k[w'(\xi_2) - w'(\xi_3)]$$

We will work out the differences in the square brackets using Equations (18) and (27), and also the constants determined above which enter the solution. Finally, if we compare the pressure jump in the reflected wave with that in the converging wave determined by Formula (16) to find their ratios for the same value of r, we obtain

$$\sigma = \frac{\Delta p}{(\Delta p)_0} = \frac{2}{\pi}(\mu - 1)\cos\frac{\pi\nu}{2+s} > 0 \tag{29}$$

The ratio of the velocity jumps has the same value.

It is evident from Formula (29) that the reflected wave, just like the converging wave, is a compression wave and their intensities are of the same order of magnitude in r, which is of a higher order than the excess pressure $p - p_0$ on the reflected front. The behavior of p close to the reflected front at instant t close to 0 is depicted in Fig. 3.

Notice that if in our solution we assume $C > 0$ and therefore $\lambda < 0$, $\mu < 0$, we formally arrive at the case where the converging wave is a rarefaction wave. The solution of (28) likewise proves to be unique, whilst $\sigma < 0$ (a rarefaction wave reflects as a compression wave). If we eliminate μ from Equations (28) and (29), it is possible to obtain an equation for σ:

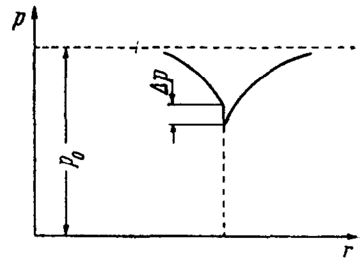

Fig. 3.

$$\frac{\pi}{2}\sec\frac{\pi\nu}{2+s}\,\sigma\tanh\left[\frac{\pi}{2}\sec\frac{\pi\nu}{2+s}\left(\sigma-\sin\frac{\pi\nu}{2+s}\right)\right]=1$$

For any value of $s > 2(\nu - 1)$ this equation possesses two roots: a positive one σ_+ (for $C < 0$) and a negative one σ_- (for $C > 0$); moreover, both roots increase monotonically with increase in $\pi\nu/(2+s)$ and $|\sigma_-| \leqslant \sigma_+ \leqslant 1$. We give values of σ_+ and of σ_- for several values of $\nu^* = [\pi\nu/(2+s)]°$:

$\nu^* =$	0	15	30	45	60	75	90
$\sigma_+ =$	0.764	0.837	0.896	0.941	0.974	0.993	1.0
$\sigma_- =$	−0.764	−0.676	−0.574	−0.455	−0.319	−0.165	0.0

Thus in the case of reflection of converging waves in a variable density gas with $s > 2(\nu - 1)$, the motion close to $r = 0$, $t = 0$ is a weak displacement. The above discussion shows, however, that several important quantitative relations and qualitative features can only be established by taking into consideration nonlinear terms in the equation.

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